

# Chapter 3

## Heat Kernels

In this chapter, we assume that the manifold  $M$  is compact and the generalized Laplacian  $H$  is not necessarily symmetric.

Note that if  $M$  is non-compact and  $H$  is symmetric, we can study the heat kernel following the lines of Spectral theorem and the Schwartz kernel theorem. We will not discuss it here.

### 3.1 heat kernels

#### 3.1.1 What is kernel?

Let  $H$  be a generalized Laplacian on a vector bundle  $E$  over a compact Riemannian oriented manifold  $M$ .

Let  $E$  and  $F$  be Hermitian vector bundles over  $M$ . Let  $p_1$  and  $p_2$  be the projections from  $M \times M$  onto the first and the second factor  $M$  respectively. We denote by

$$E \boxtimes F := p_1^*E \otimes p_2^*F \quad (3.1.1)$$

over  $M \times M$ .

**Definition 3.1.1.** A continuous section  $P(x, y)$  on  $F \boxtimes E^*$  is called a kernel. Using  $P(x, y)$ , we could define an operator  $P : \mathcal{C}^\infty(M, E) \rightarrow \mathcal{C}^\infty(M, F)$  by

$$(Pu)(x) = \int_{y \in M} \langle P(x, y), u(y) \rangle_E dy. \quad (3.1.2)$$

The kernel  $P(x, y)$  is also called the kernel of  $P$ , which is also denoted by  $\langle x|P|y \rangle$ .

**Proposition 3.1.2.** *If  $P$  has a kernel  $P(x, y)$ , then the adjoint operator  $P^*$  has a kernel  $P^*(x, y) = P(y, x)^* \in \mathcal{C}^\infty(M \times M, E^* \boxtimes F)^1$ .*

*Proof.* For  $u \in L^2(M, E)$ ,  $v \in L^2(M, F^*)$ , we have

$$\begin{aligned} (Pu, v)_{L^2} &= \int_{x \in M} \left\langle \int_{y \in M} \langle P(x, y), u(y) \rangle_E dy, v(x) \right\rangle_F dx \\ &= \int_{y \in M} \left\langle u(y), \int_{x \in M} \langle P(x, y)^*, v(x) \rangle_F dx \right\rangle_E dy \\ &= \int_{x \in M} \left\langle u(x), \int_{y \in M} \langle P(y, x)^*, v(y) \rangle_F dy \right\rangle_E dx = (u, P^*v)_{L^2}. \end{aligned} \quad (3.1.3)$$

So for any  $v \in L^2(M, F^*)$ ,

$$P^*v = \int_{y \in M} \langle P(y, x)^*, v(y) \rangle_F dy. \quad (3.1.4)$$

The proof of Proposition 3.1.2 is completed.  $\square$

**Proposition 3.1.3.** *If  $P$  has a smooth kernel, then  $P$  is a smoothing operator.*

*Proof.* For any  $m \in \mathbb{R}$ ,  $\alpha \in \mathbb{N}^n$ ,  $u \in \mathcal{C}^\infty(M, E)$ , from Theorem 1.2.33 (3), we have

$$|D_x^\alpha Pu(x)| = \left| \int_{y \in M} \langle D_x^\alpha P(x, y), u(y) \rangle_E dy \right| \leq \|D_x^\alpha P(x, y)\|_{y, m} \|u\|_{-m}. \quad (3.1.5)$$

So for any  $s, m \in \mathbb{N}$ ,  $u \in \mathcal{C}^\infty(M, E)$ ,

$$\|Pu\|_s \leq \sum_{|\alpha| \leq s} \left\| \|D_x^\alpha P(x, y)\|_{y, m} \right\|_x \|u\|_{-m}. \quad (3.1.6)$$

Since  $P(x, y)$  is smooth on  $x$  and  $y$ , from (1.2.91),  $\left\| \|D_x^\alpha P(x, y)\|_{y, m} \right\|_x$  is uniformly bounded. So  $P$  is a smoothing operator.

The proof of Proposition 3.1.3 is completed.  $\square$

<sup>1</sup>From (3.1.1),  $F \boxtimes E^* = \text{Hom}(p_1^*E, p_2^*F)$ . For  $A \in F \boxtimes E^* = \text{Hom}(p_1^*E, p_2^*F)$ , we take  $A^* \in \text{Hom}(p_2^*F, p_1^*E) = E^* \boxtimes F$  as the transpose of the matrix.

### 3.1.2 Heat kernel for symmetric generalized Laplacian

Now we study the solution of the heat equation on manifold with initial condition:

$$\begin{cases} \left(\frac{\partial}{\partial t} + H\right) u(t, x) = 0, & t > 0, \\ \lim_{t \rightarrow 0} u(t, x) = u(x) \in L^2(M, E). \end{cases} \quad (3.1.7)$$

If  $H$  is symmetric, from Theorem 2.2.47, there exists a complete orthonormal basis  $\{\varphi_i\} \subset L^2(M, E)$  such that  $H\varphi_i = \lambda_i\varphi_i$ , where  $\{\lambda_i\} = \sigma(H)$ . In this case, if  $u = \varphi_i$ , then

$$u(t, x) = e^{-t\lambda_i}u(x) \quad (3.1.8)$$

is the unique solution of (3.1.7). In general, for  $u = \sum_i a_i\varphi_i \in L^2(M, E)$ , the unique solution of (3.1.7) is

$$u(t, x) = \sum_i a_i e^{-t\lambda_i} \varphi_i(x). \quad (3.1.9)$$

Inspired of (3.1.8), for  $t > 0$ , we define the **heat operator**  $e^{-tH} : L^2(M, E) \rightarrow L^2(M, E)$  by

$$u(t, x) = e^{-tH}u(x). \quad (3.1.10)$$

For any  $s, t > 0$ , from (3.1.9) and (3.1.10), we have

$$e^{-tH}e^{-sH} = e^{-(s+t)H}, \quad (3.1.11)$$

which means that the heat operators form a semi-group.

Let

$$e^{-tH}(x, y) = \sum_i e^{-t\lambda_i} \varphi_i(x) \boxtimes \varphi_i(y)^*. \quad (3.1.12)$$

Then from (3.1.9)-(3.1.12), formally,

$$\begin{aligned} e^{-tH}u(x) &= \sum_i a_i e^{-t\lambda_i} \varphi_i(x) = \sum_i a_i e^{-t\lambda_i} \varphi_i(x) \int_{y \in M} \left\langle \varphi_i(y)^*, \sum_j a_j \varphi_j(y) \right\rangle dy \\ &= \int_{y \in M} \left\langle \sum_i e^{-t\lambda_i} \varphi_i(x) \boxtimes \varphi_i(y)^*, \sum_j a_j \varphi_j(y) \right\rangle dy \\ &= \int_{y \in M} \langle e^{-tH}(x, y), u(y) \rangle dy. \end{aligned} \quad (3.1.13)$$

So if (3.1.12) is uniformly convergent in  $\mathcal{C}^r$ -norm for any  $r \in \mathbb{N}$ , i.e.,  $e^{-tH}(x, y)$  is smooth on  $x$  and  $y$ ,  $e^{-tH}(x, y)$  is the smooth kernel of the heat operator  $e^{-tH}$ , called the **heat kernel**. In the followings, we will prove that (3.1.12) is uniformly convergent in  $\mathcal{C}^r$ -norm for any  $r \in \mathbb{N}$ .

**Lemma 3.1.4.** *Let  $P$  be a self-adjoint elliptic differential operator of order  $m > 0$  on Hermitian vector bundle  $E$  over compact Riemannian manifold  $M$ . If we order the eigenvalues  $|\lambda_1| \leq |\lambda_2| \leq \dots$ , then*

(1) *for any  $l \in \mathbb{N}$ , there exists  $C_l > 0$  such that for  $k \in \mathbb{N}$ ,  $s > (2l+n)/2m$ ,  $s \in \mathbb{N}$ , we have*

$$\|\varphi_k\|_{\mathcal{C}^l} \leq C_l(1 + |\lambda_k|^s), \quad (3.1.14)$$

where  $\varphi_k$  is the eigenfunction with respect to  $\lambda_k$  such that  $\|\varphi_k\|_{L^2} = 1$ ;

(2) *there exists  $C > 0$  and  $\varepsilon > 0$  such that for any  $k \in \mathbb{N}$ ,*

$$|\lambda_k| \geq Ck^\varepsilon. \quad (3.1.15)$$

*Proof.* (1) From the Sobolev embedding theorem and the elliptic estimate, there exist  $C'_l, C' > 0$  such that

$$\|\varphi_k\|_{\mathcal{C}^l} \leq C'_l \|\varphi_k\|_{ms} \leq C'_l C' (\|\varphi_k\|_0 + \|P^s \varphi_k\|_0) = C'_l C' (1 + |\lambda_k|^s). \quad (3.1.16)$$

(2) If we replace  $P$  by  $P^s$ , we replace  $\lambda_i$  by  $\lambda_i^s$ . So we only need to prove (2) for  $m > n/2$ . Set

$$F(k) := \text{span}_{i \leq k} \{\varphi_i\} \subset \mathcal{C}^\infty(M, E). \quad (3.1.17)$$

From Sobolev embedding theorem and the elliptic estimate, for  $\varphi \in F(k)$ , for any  $x \in M$ , we have

$$|\varphi(x)| \leq C \|\varphi\|_m \leq C (\|P\varphi\|_0 + \|\varphi\|_0) \leq C(1 + |\lambda_k|) \|\varphi\|_0. \quad (3.1.18)$$

Let  $\varphi = \sum_j c_j \varphi_j$ . Then

$$\left| \sum_{1 \leq j \leq k} c_j \varphi_j(x) \right| = |\varphi(x)| \leq C(1 + |\lambda_k|) \left( \sum_j |c_j|^2 \right)^{1/2}. \quad (3.1.19)$$

Choose a local orthonormal frame of  $E$  and decompose  $\varphi_j(x)$  into components  $\varphi_{\nu,j}$  for  $1 \leq \nu \leq p$ , where  $\dim E = p$ . Then

$$|\varphi(x)|^2 = \sum_{1 \leq \nu \leq p} \left| \sum_{1 \leq j \leq k} c_j \varphi_{\nu,j}(x) \right|^2. \quad (3.1.20)$$

We fix  $\nu$  and take  $c_j = \varphi_{\nu,j}^*(x)$ . Then from (3.1.19) and (3.1.20), we have

$$\sum_{1 \leq j \leq k} |\varphi_{\nu,j}(x)|^2 \leq C(1 + |\lambda_k|) \left( \sum_j |\varphi_{\nu,j}(x)|^2 \right)^{1/2}. \quad (3.1.21)$$

So

$$\sum_{1 \leq j \leq k} |\varphi_{\nu,j}(x)|^2 \leq C^2(1 + |\lambda_k|)^2. \quad (3.1.22)$$

Sum over  $\nu$ , we have

$$\sum_{1 \leq j \leq k} |\varphi_j(x)|^2 \leq pC^2(1 + |\lambda_k|)^2. \quad (3.1.23)$$

Integral (3.1.23) over  $M$ , we have

$$k \leq pC^2 \text{vol}(M)(1 + |\lambda_k|)^2. \quad (3.1.24)$$

So we obtain (2).

The proof of Lemma 3.1.4 is completed.  $\square$

**Proposition 3.1.5.** *The heat kernel exists if  $H$  is symmetric. More precisely, (3.1.12) is uniformly convergent in  $\mathcal{C}^r$ -norm for any  $r \in \mathbb{N}$ , i.e.,  $e^{-tH}(x, y)$  is smooth on  $x$  and  $y$ .*

*Proof.* Since  $H$  is bounded from below, there are only finitely negative eigenvalues. Since we consider the convergence when  $i \rightarrow \infty$ , we may assume that  $H > 0$ . From Lemma 3.1.4, for  $s > (l + l' + n/m)$ ,  $s \in \mathbb{N}$ ,

$$\|e^{-t\lambda_k} \varphi_k(x) \boxtimes \varphi_k(y)^*\|_{\mathcal{C}_x^l, \mathcal{C}_y^{l'}} \leq C_l C_{l'} e^{-t\lambda_k} (1 + \lambda_k)^s. \quad (3.1.25)$$

From (3.1.15), for any  $s' \leq s$ ,  $s \in \mathbb{N}$ , we have

$$e^{-t\lambda_k} \lambda_k^{s'} \leq C_k t^{-s'} e^{-t\lambda_k/2} \leq C_l t^{-s'} e^{-tCk^\varepsilon}. \quad (3.1.26)$$

From (3.1.25) and (3.1.26), we have

$$\sum_k \|e^{-t\lambda_k} \varphi_k(x) \boxtimes \varphi_k(y)^*\|_{\mathcal{C}_x^l, \mathcal{C}_y^{l'}} \leq C t^{-s'} \sum_k e^{-tCk^\varepsilon} < +\infty. \quad (3.1.27)$$

Remark that the convergence of  $\sum_k e^{-Ck^\varepsilon}$  is equivalent to that of  $\int_0^{+\infty} e^{-Cx^\varepsilon} dx = C^{-\varepsilon^{-1}} (\varepsilon^{-1} - 1) \int_0^{+\infty} y^{\varepsilon^{-1}-1} e^{-y} dy$ , which is a Gamma function.

The proof of Proposition 3.1.5 is completed.  $\square$

Note that from (3.1.7) and (3.1.13), for any  $u(x) \in L^2(M, E)$ ,

$$\left(\frac{\partial}{\partial t} + H\right)(e^{-tH}u)(x) = \int_{y \in M} \left(\frac{\partial}{\partial t} + H_x\right) e^{-tH}(x, y)u(y)dy = 0. \quad (3.1.28)$$

So we have

$$\left(\frac{\partial}{\partial t} + H_x\right) e^{-tH}(x, y) = 0. \quad (3.1.29)$$

### 3.1.3 Heat kernel on Euclidean space

Let  $M = \mathbb{R}$ ,  $E = \mathbb{C}$  and  $H = \Delta = -\frac{d^2}{dx^2}$ . From the knowledge of the PDE course for undergraduates, the heat kernel is

$$e^{-t\Delta}(x, y) = \frac{1}{(4\pi t)^{\frac{1}{2}}} e^{-\frac{(x-y)^2}{4t}}. \quad (3.1.30)$$

Thus for  $u \in L^2(\mathbb{R})$ ,

$$(e^{-t\Delta}u)(x) = \frac{1}{(4\pi t)^{\frac{1}{2}}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4t}} u(y)dy. \quad (3.1.31)$$

**Proposition 3.1.6.** *For even  $l \in \mathbb{N}$ , if  $\|u\|_{\mathcal{C}^{l+1}} \leq +\infty$ , then there exists  $C > 0$  such that*

$$\left| e^{-t\Delta}u - \sum_{k=0}^{l/2} \frac{(-t)^k}{k!} \Delta^k u \right| \leq Ct^{l/2+1}. \quad (3.1.32)$$

*This is another explanation why we write heat operator as  $e^{-tH}$ .*

*Proof.* From (3.1.31),

$$(e^{-t\Delta}u)(x) = \frac{1}{(4\pi)^{\frac{1}{2}}} \int_{\mathbb{R}} e^{-\frac{v^2}{4}} u(x + \sqrt{t}v)dv. \quad (3.1.33)$$

By Taylor expansion,

$$u(x + \sqrt{t}v) = \sum_{k=0}^l \frac{(\sqrt{t}v)^k}{k!} u^{(k)}(x) + \frac{(\sqrt{t}v)^{l+1}}{l!} \int_0^1 (1-s)^l u^{(l+1)}(x + s\sqrt{t}v)ds. \quad (3.1.34)$$

Since  $\|u\|_{\mathcal{C}^{l+1}} \leq +\infty$ ,

$$\left| u(x + \sqrt{t}v) - \sum_{k=0}^l \frac{t^{k/2} v^k}{k!} u^{(k)}(x) \right| \leq \frac{t^{(l+1)/2} v^{l+1}}{l!} \|u\|_{l+1}. \quad (3.1.35)$$

From (3.1.33) and (3.1.35), we have

$$\begin{aligned} \left| (e^{-t\Delta} u)(x) - \sum_{k=0}^l \frac{1}{(4\pi)^{\frac{1}{2}}} \int_{\mathbb{R}} e^{-\frac{v^2}{4}} v^k dv \cdot \frac{t^{k/2}}{k!} u^{(k)}(x) \right| \\ \leq \frac{1}{(4\pi)^{\frac{1}{2}}} \int_{\mathbb{R}} e^{-\frac{v^2}{4}} v^{l+1} dv \cdot \frac{t^{(l+1)/2}}{l!} \|u\|_{l+1}. \end{aligned} \quad (3.1.36)$$

Let

$$A(t) = \sum_{k=0}^{+\infty} \frac{1}{(4\pi)^{\frac{1}{2}}} \frac{t^k}{k!} \int_{\mathbb{R}} e^{-\frac{v^2}{4}} v^k dv = \frac{1}{(4\pi)^{\frac{1}{2}}} \int_{\mathbb{R}} e^{-\frac{v^2}{4} + tv} dv. \quad (3.1.37)$$

Let  $u = v - 2t$ . We have

$$A(t) = \frac{1}{(4\pi)^{\frac{1}{2}}} \int_{\mathbb{R}} e^{-\frac{u^2}{4} + t^2} du = e^{t^2}. \quad (3.1.38)$$

So we have

$$\frac{1}{(4\pi)^{\frac{1}{2}}} \int_{\mathbb{R}} e^{-\frac{v^2}{4}} v^k dv = \begin{cases} \frac{k!}{(k/2)!}, & \text{if } k \text{ even;} \\ 0, & \text{if } k \text{ odd.} \end{cases} \quad (3.1.39)$$

From (3.1.36) and (3.1.39), we obtain (3.1.32).

The proof of Proposition 3.1.6 is completed.  $\square$

### 3.1.4 Non-symmetric heat kernel

In this subsection, the generalized Laplacian may be non-symmetric.

Compare with the symmetric case, we define the heat kernel by summarizing the properties that the kernel of an operator  $e^{-tH}$  must have.

**Definition 3.1.7.** A heat kernel for  $H$  is a continuous section  $e^{-tH}(x, y)$  of the bundle  $E \boxtimes E^*$  over  $\mathbb{R}_+ \times M \times M$ , satisfying the following properties:

(1)  $e^{-tH}(x, y)$  is  $\mathcal{C}^1$  with respect to  $t$  and  $\mathcal{C}^2$  with respect to  $x$ , i.e.,  $\frac{\partial}{\partial t} e^{-tH}(x, y)$  is continuous and  $\frac{\partial^2}{\partial x_i \partial x_j} e^{-tH}(x, y)$  are continuous for any coordinate system of  $x$ ;

(2)  $(\frac{\partial}{\partial t} + H_x) e^{-tH}(x, y) = 0$ ;

(3) Let  $e^{tH}$  be the operator defined as in (3.1.2), called the **heat operator**. Then for any  $s \in \mathcal{C}^\infty(M, E)$ ,  $\lim_{t \rightarrow 0} e^{-tH} s = s$  with respect to the  $\mathcal{C}^0$ -norm.

We need to prove that the heat kernel in Definition 3.1.7 exists and is unique. We first assume the existence and study the uniqueness. In the next section, we will prove that for any generalized Laplacian, the heat kernel always exists and smooth on  $t, x, y$ .

**Lemma 3.1.8.** *Assume that  $H^*$  has a heat kernel. If  $s(t, x)$  is a map from  $\mathbb{R}_+$  to the space of sections of  $E$  which is  $\mathcal{C}^1$  in  $t$  and  $\mathcal{C}^2$  in  $x$  (in the meaning of Definition 3.1.7 (1)), such that  $\lim_{t \rightarrow 0} s(t, x) = 0$  and which satisfies the heat equation  $(\frac{\partial}{\partial t} + H_x)s(t, x) = 0$ , then  $s(t, x) = 0$*

*Proof.* For any  $u \in \mathcal{C}^\infty(M, E^*)$ ,  $0 < \theta < t$ , let

$$f(\theta) = \int_{M \times M} \langle s(\theta, x), e^{-(t-\theta)H^*}(x, y)u(y) \rangle dx dy. \quad (3.1.40)$$

From the heat equation in Definition 3.1.7, we have

$$\begin{aligned} \frac{\partial}{\partial \theta} f(\theta) &= \int_{M \times M} \left\langle \frac{\partial}{\partial \theta} s(\theta, x), e^{-(t-\theta)H^*}(x, y)u(y) \right\rangle dx dy \\ &\quad + \int_{M \times M} \left\langle s(\theta, x), \frac{\partial}{\partial \theta} e^{-(t-\theta)H^*}(x, y)u(y) \right\rangle dx dy \\ &= \int_{M \times M} \langle -H_x s(\theta, x), e^{-(t-\theta)H^*}(x, y)u(y) \rangle dx dy \\ &\quad + \int_{M \times M} \langle s(\theta, x), H_x^* e^{-(t-\theta)H^*}(x, y)u(y) \rangle dx dy = 0. \end{aligned} \quad (3.1.41)$$

Since  $\lim_{\theta \rightarrow 0} s(\theta, x) = 0$ ,  $\lim_{\theta \rightarrow 0} f(\theta) = 0$ . So

$$0 = f(t) = \int_{M \times M} \langle s(t, x), u(y) \rangle dx dy \quad (3.1.42)$$

for any  $u \in \mathcal{C}^\infty(M, E^*)$ . Thus for any  $t > 0$ ,  $s(t, x) = 0$ .

The proof of Lemma 3.1.8 is completed.  $\square$

**Proposition 3.1.9.** (1) *If  $H^*$  has a heat kernel, then  $H$  has at most one heat kernel.*

(2) *If  $H$  and  $H^*$  have heat kernels, then*

$$e^{-tH^*}(x, y) = e^{-tH}(y, x)^*. \quad (3.1.43)$$

(3) *If  $H$  and  $H^*$  have heat kernels, then  $\{e^{-tH}\}_{t>0}$  form a semi-group.*

*Proof.* For any  $u, s \in \mathcal{C}^\infty(M, E^*)$ ,  $0 < \theta < t$ , let

$$f(\theta) = \int_M \langle e^{-\theta H} s(x), e^{-(t-\theta)H^*} u(x) \rangle dx. \quad (3.1.44)$$

As in (3.1.41),  $f'(\theta) = 0$ . So

$$(e^{-tH} s, u)_{L^2} = f(t) = \lim_{t \rightarrow 0} f(0) = (s, e^{-tH^*} u)_{L^2}. \quad (3.1.45)$$

So

$$\exp(-tH^*) = \exp(-tH)^*. \quad (3.1.46)$$

From Proposition 3.1.2, we get (1) and (2).

For (3), set

$$s_t = e^{-tH} e^{-\theta H} s - e^{-(t+\theta)H} s. \quad (3.1.47)$$

Then  $(\partial_t + H)s_t = 0$ . Since  $\lim_{t \rightarrow 0} s_t = 0$ , by Lemma 3.1.8,  $s_t = 0$  for any  $t > 0$ .

The proof of Proposition 3.1.9 is completed.  $\square$

**Proposition 3.1.10.** *Assume that  $H$  and  $H^*$  have heat kernels. Let  $\Delta$  be a connected Cauchy domain with  $\Gamma = \partial\Delta$  such that  $\lim_{t \rightarrow \pm\infty} \operatorname{Re}(\Gamma(t)) = +\infty$ . Assume that there exists a polynomial  $g \in \mathbb{C}[z]$  such that for any  $\lambda \notin \Delta$ ,  $\|(\lambda - H)^{-1}\| \leq g(\lambda)$ . Then the heat operator*

$$e^{-tH} = \frac{1}{2\pi i} \int_{\Gamma} e^{-t\lambda} (\lambda - H)^{-1} d\lambda. \quad (3.1.48)$$

*So our notation  $e^{-tH}$  is compatible with that in functional calculus.*

*Proof.* Note that the operators on both sides of (3.1.48) are bounded for  $t > 0$ . Let

$$f(t) = \frac{1}{2\pi i} \int_{\Gamma} e^{-t\lambda} (\lambda - H)^{-1} d\lambda. \quad (3.1.49)$$

Then we have

$$Hf(t) = f(t)H. \quad (3.1.50)$$

By Cauchy integral formula and  $\lim_{t \rightarrow \pm\infty} \operatorname{Re}(\Gamma(t)) = +\infty$ ,  $\int_{\Gamma} e^{-t\lambda} d\lambda = 0$ . Since  $\lambda(\lambda - H)^{-1} = (\lambda - H)^{-1}H + \operatorname{Id}$ , for any  $s \in \mathcal{C}^\infty(M, E)$ ,

$$\begin{aligned} \frac{\partial}{\partial t} f(t)s &= \frac{1}{2\pi i} \int_{\Gamma} \frac{\partial}{\partial t} e^{-t\lambda} (\lambda - H)^{-1} s d\lambda = -\frac{1}{2\pi i} \int_{\Gamma} \lambda e^{-t\lambda} (\lambda - H)^{-1} s d\lambda \\ &= -\frac{1}{2\pi i} \int_{\Gamma} e^{-t\lambda} (\lambda - H)^{-1} H s d\lambda = -f(t)Hs = -Hf(t)s. \end{aligned} \quad (3.1.51)$$

So for any  $s \in \mathcal{C}^\infty(M, E)$ ,  $f(t)s$  satisfies the heat equation.

For  $\theta, t > 0$ , since  $f(\theta)e^{-tH}s$  and  $e^{-tH}f(\theta)s$  satisfy the heat equation and  $\lim_{t \rightarrow 0} f(\theta)e^{-tH}s = \lim_{t \rightarrow 0} e^{-tH}f(\theta)s = f(\theta)s$ , by Lemma 3.1.8, we have

$$f(\theta)e^{-tH} = e^{-tH}f(\theta). \quad (3.1.52)$$

From functional calculus Theorem 2.3.17, for  $\theta, t > 0$ , we have

$$f(\theta + t) = f(\theta)f(t) = f(t)f(\theta). \quad (3.1.53)$$

From (3.1.53),  $\lim_{t \rightarrow 0} f(t)f(\theta)s = f(\theta)s$ . So from Lemma 3.1.8 again, we have

$$f(t)f(\theta) = e^{-tH}f(\theta). \quad (3.1.54)$$

By (3.1.52)-(3.1.54), for  $t > 0$  fixed and any  $\theta > 0$ ,

$$f(\theta)(e^{-tH} - f(t)) = 0. \quad (3.1.55)$$

Taking  $\theta \rightarrow 0$ , we obtain (3.1.48).

The proof of Proposition 3.1.10 is completed.  $\square$